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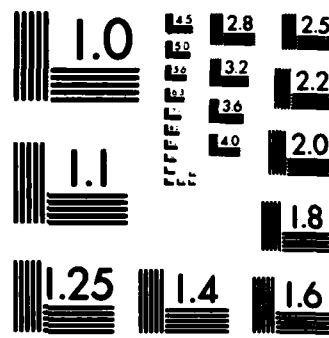
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**CONVERGENCE OF QUASI-STATIONARY TO STATIONARY
DISTRIBUTIONS FOR STOCHASTICALLY
MONOTONE MARKOV PROCESSES**

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by

**Moshe Pollak
Hebrew University of Jerusalem**

and

**David Siegmund
Stanford University**

**TECHNICAL REPORT NO. 31
OCTOBER 1984**

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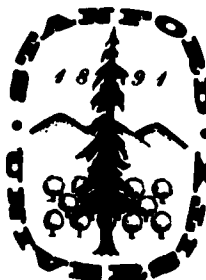
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Convergence of Quasi-Stationary to Stationary Distributions for Stochastically Monotone Markov Processes

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Abstract. It is shown that if a stochastically monotone Markov process on $[0, \infty)$ with stationary distribution H has its state space truncated by making all states in $[B, \infty)$ absorbing, then the quasi-stationary distribution of the new process converges to H as $B \rightarrow \infty$. *infinity*

1. Introduction.

Let $X(t)$, $0 \leq t < \infty$, be a Markov process with state space $[0, \infty)$. If the process is absorbing, its quasi-stationary distribution is defined to be the limit (if it exists) as $t \rightarrow \infty$ of the distribution of $X(t)$ given that absorption has not occurred by the time t (e.g. Seneta and Vere-Jones, 1966). We shall consider a process which in fact has a limiting probability distribution H in the sense that for every initial state $X(0) = x$,

$$\lim_{t \rightarrow \infty} P^x\{X(t) < y\} = H(y) \quad (1)$$

at continuity points y of H , but which is made into an absorbing process by introduction of the stopping time

$$\tau = \tau_B = \inf\{t : X(t) \geq B\} \quad (B > 0). \quad (2)$$

The absorbing process $X(t \wedge \tau)$, $0 \leq t < \infty$, may have a quasi-stationary distribution

$$\lim_{t \rightarrow \infty} P^x\{X(t) < y \mid \tau > t\},$$

and a natural question is whether this distribution converges to H as $B \rightarrow \infty$. More

generally, we shall be interested in establishing under some conditions that as $t, B \rightarrow \infty$

$$P^z\{X(t) < y \mid \tau > t\} \rightarrow H(y) \quad (3)$$

for all z and continuity points y of H .

Our principal result, Theorem 1, asserts that (3) holds if the process $X(t)$ is stochastically monotone. (See Section 2 for the definition.) We also ask whether the hypothesis of stochastic monotonicity implies the condition (1) and show that under a mild condition of communicability it does (Theorem 3), although the limit H obviously need not be a probability distribution.

Seneta (1980), Section 7.3, in the special case of Markov chains, has studied the convergence of quasi-stationary distributions (assumed to exist) for a truncated state space to the stationary distribution of the original chain. His methods and results are completely different than ours.

Our motivation to consider the problem discussed here comes from our efforts to compare competing stopping rules in a quickest detection problem (Pollak and Siegmund, 1984). There we were concerned with two diffusion processes on $[0, \infty)$, which "detect" a change in the drift of an underlying Brownian motion process if they ever reach a given high level B . We were interested in evaluating the expected delay from the time that the change actually occurs until it is detected, given that the change has not been falsely detected before its occurrence. If the change occurs after the process has been running for some time, we must evaluate the expected first passage of our (changed) diffusion processes to the level B , given that they start in the quasi-stationary distribution. Since the quasi-stationary distribution is rather difficult to evaluate in one case, and since B is almost always quite large, it is natural to consider the quasi-stationary distribution as $B \rightarrow \infty$, or more generally the convergence indicated in (3).

2. Results.

Let $X_n, n = 0, 1, 2, \dots$ be a Markov process in discrete time with state space $[0, \infty)$ and stationary transition probabilities. We write P^z to denote probability for the process with initial state $X_0 = z$.



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Special

We shall call the process stochastically monotone if $P^z\{X_1 \geq y\}$ is nondecreasing and right continuous in z for all y . (Right continuity is not usually included in this definition and for some purposes is easily eliminated. Since it seems harmless in applications, we have not tried to achieve the utmost generality.)

Theorem 1. Let X_n , $n = 0, 1, 2, \dots$ be stochastically monotone, and let τ be defined by (2). For arbitrary $x, y, B \geq 0$ and $m = 1, 2, \dots$

$$P^z\{X_m < y \mid \tau > m\} \geq P^z\{X_m < y\}. \quad (4)$$

If (1) is also satisfied, then as $B, m \rightarrow \infty$, for arbitrary z

$$P^z\{X_m < y \mid \tau_B > m\} \rightarrow H(y) \quad (5)$$

at all continuity points y of H .

Proof. We begin by proving (4), which is trivial for $m = 1$. Suppose it is true for $m = n$, and consider the case $m = n + 1$,

$$\begin{aligned} P^z\{X_{n+1} < y \mid \tau > n + 1\} &= \frac{P^z\{X_{n+1} < y \mid \tau > n\}}{P^z\{X_{n+1} < B \mid \tau > n\}} \\ &\geq \int_{[0, B)} P^z\{X_n \in dz \mid \tau > n\} P^z\{X_1 < y\} \end{aligned} \quad (6)$$

To perform the following manipulations it is convenient (although not necessary) to introduce the dual probability \tilde{P} in the sense of Siegmund (1976), under which $\{X_n, n = 0, 1, \dots\}$ is again a Markov chain and satisfies

$$P^z\{X_n \geq y\} = \tilde{P}^z\{X_n \leq z\}$$

for all $x, y \in [0, \infty) \times [0, \infty)$. Then (6) can be continued as follows

$$\begin{aligned}
&= \int_{[0, B)} P^x \{X_n \in dz \mid \tau > n\} \tilde{P}^y \{X_1 < z\} \\
&= \int_{[0, B)} P^x \{X_n \in dz \mid \tau > n\} \int_{[0, z)} \tilde{P}^y \{X_1 \in d\xi\} \\
&= \int_{[0, \infty)} \tilde{P}^y \{X_1 \in d\xi\} \int_{[0, B \wedge \xi)} P^x \{X_n \in dz \mid \tau > n\} \\
&= \int_{[0, \infty)} \tilde{P}^y \{X_1 \in d\xi\} P^x \{X_n < B \wedge \xi \mid \tau > n\} \\
&\geq \int_{[0, \infty)} \tilde{P}^y \{X_1 \in d\xi\} P^x \{X_n < \xi\}
\end{aligned} \tag{7}$$

The inequality follows from the inductive hypothesis if $\xi < B$ and because the conditional probability equals 1 if $\xi \geq B$. Hence the last expression in (7) equals

$$\begin{aligned}
&\int_{[0, \infty)} \tilde{P}^y \{X_1 \in d\xi\} \int_{[0, \xi)} P^x \{X_n \in dz\} = \int_{[0, \infty)} P^x \{X_n \in dz\} \tilde{P}^y \{X_1 > z\} \\
&= \int_{[0, \infty)} P^x \{X_n \in dz\} P^x \{X_1 < y\} = P^x \{X_{n+1} < y\},
\end{aligned}$$

which completes the proof of (4).

From (4) and the hypothesis (1) it follows that

$$\liminf_{m, B} P^x \{X_m < y \mid \tau_B > m\} \geq H(y).$$

Hence it remains to prove the reverse inequality.

Let $\epsilon > 0$ and fix k so large that

$$P^0 \{X_k < y\} \leq H(y) + \epsilon.$$

For arbitrary $0 < \eta < 1$, by (4) and hypothesis (1) there exists $\gamma > 0$ such that for all n, B

$$P^x \{X_n < \gamma \mid \tau_B > n\} \geq P^x \{X_n < \gamma\} \geq 1 - \eta.$$

Also for any $0 < \delta < 1$ there exists B_0 such that for all $B \geq B_0$

$$P^\gamma \{\tau_B > k\} \geq 1 - \delta.$$

Hence for all $m > k$ and $B \geq B_0$

$$\begin{aligned}
P^x\{X_m < y \mid r > m\} &= P^x\{X_m < y \mid r > m - k, r > m\} \\
&= P^x\{r > m, X_m < y \mid r > m - k\} / P^x\{r > m \mid r > m - k\} \\
&\leq \frac{\int_{[0, B)} P^x\{X_{m-k} \in dz \mid r > m - k\} P^x\{X_k < y\}}{\int_{[0, \gamma)} P^x\{X_{m-k} \in dz \mid r > m - k\} P^x\{r > k\}} \\
&\leq P^0\{X_k < y\} / P^x\{X_{m-k} < \gamma \mid r > m - k\} P^\gamma\{r > k\} \\
&\leq [H(y) + \epsilon] / (1 - \eta)(1 - \delta).
\end{aligned}$$

Since ϵ , η , and δ are arbitrary this completes the proof.

It is straightforward to extend Theorem 1 to the case of a continuous time Markov process. In fact, it is only the proof of (4) that uses the discreteness of the time scale; and this inequality is easily extended to continuous time by a limiting process.

More precisely, the Markov process $X(t)$, $0 \leq t < \infty$, with stationary transitions is called stochastically monotone if for some $h > 0$ and for all $0 < t < h$, $P^x\{X(t) \geq y\}$ is nondecreasing and right continuous in x for all y . Hence for any $0 < \epsilon < h$ the discrete skeleton $X(n\epsilon)$, $n = 0, 1, \dots$, is stochastically monotone, so equation (4) holds for it. If we now add a smoothness assumption on the sample paths, say that they are right continuous, we obtain a version of (4) by passing to the limit as $\epsilon \rightarrow 0$. For completeness we summarize these results in the following theorem.

Theorem 2. Let $X(t)$, $0 < t < \infty$, be stochastically monotone with right continuous sample paths, and let r be defined by (2). For arbitrary x, y, B , and $t \geq 0$

$$P^x\{X(t) < y \mid r > t\} \geq P^x\{X(t) < y\}.$$

If (1) is also satisfied, then as $B, t \rightarrow \infty$, for arbitrary x

$$P^x\{X(t) < y \mid r > t\} \rightarrow H(y)$$

at all continuity points y of H .

Remark. Simple coupling arguments show that birth and death and diffusion processes are stochastically monotone, so Theorem 2 applies to them. In particular it applies to the two diffusion processes which motivated our investigation. (See Lemma 1 of Pollak and Siegmund, 1984.)

We now ask whether our two basic assumptions, stochastic monotonicity and existence of a limiting distribution, are to some extent redundant. If X_n , $n = 0, 1, \dots$, (or $X(t)$, $0 \leq t < \infty$) is stochastically monotone, with state space $[0, \infty)$, it follows from Theorem 1 of Siegmund (1976) that there is a dual probability \tilde{P} under which X_n , $n = 0, 1, \dots$, is a Markov chain with stationary transition probabilities which satisfy

$$P^x\{X_n \geq y\} = \tilde{P}^y\{X_n \leq x\}. \quad (8)$$

By putting $y = 0$ we see that 0 is an absorbing state for the \tilde{P} process, so for $x = 0$

$$P^0\{X_n \geq y\} = \tilde{P}^y\{X_n = 0\} = \tilde{P}^y\left(\bigcup_{k=1}^n \{X_k = 0\}\right), \quad (9)$$

which is increasing in n and hence has some limit, say $\tilde{H}(y)$. Of course, \tilde{H} need not be a probability. The following theorem shows that under an hypothesis of communicability, $P^x\{X_n \geq y\} \rightarrow \tilde{H}(y)$ for arbitrary initial states x .

Theorem 3. Let $x > 0$. If $\lim_{n \rightarrow \infty} P^0\{X_n \geq x\} = \tilde{H}(x) > 0$, then for all y $\lim_{n \rightarrow \infty} P^x\{X_n \geq y\}$ exists and equals $\tilde{H}(y)$.

We begin with a simple lemma.

Lemma 1. Let $\tilde{\tau}_0 = \inf\{n : X_n = 0\}$. If $\limsup_n \tilde{P}^y\{\tilde{\tau}_0 > n, X_n \leq x\} > 0$, then

$$\tilde{P}^y\{\tilde{\tau} = \infty, X_n \leq x \text{ i.o.}\} > 0.$$

Proof of Lemma 1. Since

$$\{\tilde{\tau}_0 = \infty, X_n \leq x \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\tilde{\tau}_0 > n, X_n \leq x\},$$

we have

$$\begin{aligned} \tilde{P}^y\{\tilde{\tau}_0 = \infty, X_n \leq x \text{ i.o.}\} &= \lim_{k \rightarrow \infty} \tilde{P}^y\left(\bigcup_{n=k}^{\infty} \{\tilde{\tau}_0 > n, X_n \leq x\}\right) \\ &\geq \limsup_{k \rightarrow \infty} \tilde{P}^y\{\tilde{\tau}_0 > k, X_k \leq x\} > 0. \end{aligned}$$

Proof of Theorem 3. In terms of the stopping time $\tilde{\tau}_0$ of Lemma 1, the relation (8) becomes

$$\begin{aligned} P^x\{X_n \geq y\} &= \tilde{P}^y\{\tilde{\tau}_0 \leq n\} + \tilde{P}^y\{\tilde{\tau}_0 > n, X_n \leq x\} \\ &= P^x\{X_n \geq y\} + \tilde{P}^y\{\tilde{\tau}_0 > n, X_n \leq x\}, \end{aligned}$$

so it suffices to show that

$$\tilde{P}^y\{\tilde{\tau}_0 > n, X_n \leq z\} \rightarrow 0$$

as $n \rightarrow \infty$. If this is not true, then by Lemma 1

$$\tilde{P}^y\{\tilde{\tau}_0 = \infty, X_n \leq z \text{ i.o.}\} > 0. \quad (10)$$

Also, by (9) and the assumption of the Theorem, there exist $\delta > 0$ and n_0 such that for all $0 \leq \xi \leq z$,

$$\tilde{P}^\xi\{\tilde{\tau}_0 \leq n_0\} = P^0\{X_{n_0} \geq \xi\} \geq \frac{1}{2}H(z) \geq \delta. \quad (11)$$

Let $r_s^{(1)} = \inf\{n : X_n \leq z\}$ and for $N = 2, 3, \dots$ let $r_s^{(N)} = \inf\{n : n > r_s^{(N-1)} + n_0, X_n \leq z\}$.

Also let $\tilde{\tau}_0^{(N)} = \inf\{n : n \geq r_s^{(N)}, X_n = 0\}$ and put $B_N = \{\tilde{\tau}_0^{(N)} < \infty, \tilde{\tau}_0^{(N)} - r_s^{(N)} > n_0\}$.

Obviously $\{\tilde{\tau}_0 = \infty, X_n \leq z \text{ i.o.}\} \subset \bigcap_{N=1}^{\infty} B_N$, and by (11)

$$\begin{aligned} \tilde{P}^y\left(\bigcap_{N=1}^m B_N\right) &= \sum_n \int_{[0,z]} \tilde{P}^y\left(\bigcap_{N=1}^{m-1} B_N \cap \{\tilde{\tau}_s^{(m)} = n, X_n \in d\xi\}\right) \tilde{P}^\xi\{\tilde{\tau}_0 > n_0\} \\ &\leq (1 - \delta) \tilde{P}^y\left(\bigcap_{N=1}^{m-1} B_N\right) \leq \dots \leq (1 - \delta)^m. \end{aligned}$$

Hence $\tilde{P}^y\{\tilde{\tau}_0 = \infty, X_n \leq z \text{ i.o.}\} = 0$, which contradicts (10) and completes the proof.

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